TRANSIENT RESPONSE

INTRODUCTION: - As we are aware that the terminal characteristics of capacitors and inductors are governed by differential relationships. The connection of these elements with the resistors and energy sources will result in integro-differential (or) simply differential equations with constant coefficients. The solutions of these equations in time – domain gives the "**TRANSIENT RESPONSE**" of the system of equations. The time – domain response of the circuit for different test signals is almost important to synthesize (or) design electronic circuits.

Whenever a circuit is switched from one condition to another, either by a change in the applied source or a change in the circuit elements, there is a transition period during which the branch currents and element voltages change from their former values to new values. The period is called the "**TRANSIENT STATE** (or) **NATURAL RESPONSE**". After the transient period has passed, the circuit is said to be in the "**STEADY STATE** (or) **FORCED RESPONSE**". Thus, the total response of the network is the sum of its transient response and steady state response.

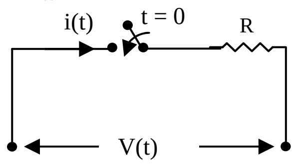
Now, the linear differential equation that describes the circuit will have two parts to its solution, the complementary function corresponding to the transient and the particular solution corresponding to the steady state.

INITIAL VALUES OF NETWORK ELEMENTS

RESISTOR: -

If a circuit is purely resistive, it does not exhibit any transient response. Thus in the circuit, the current instantaneously rises to its steady state value ie.,

$i = \frac{V}{R}$, and there is no transient response.



$t = 0^-$ $\rightarrow$ Represents the instant just prior to the closing of the switch at t = 0.

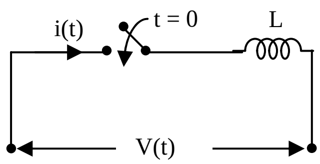
$t = 0^+$ $\rightarrow$ Represents the instant immediately after closing of the switch at t = 0.

At $t = 0^- \rightarrow i = 0$, but at $t = 0^+ \rightarrow i = \frac{V}{R}$

INDCUTOR: -

If a circuit is purely inductor as shown in figure.

If the switch is closed at t = 0, by, applying KVL to the circuit



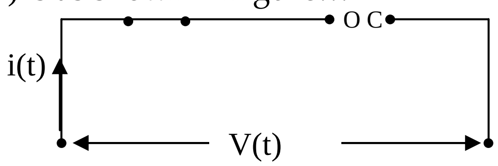
$$V = L\frac{di}{dt} \quad for\ t > 0^+$$

At $t = 0^-$, the current is zero, assuming the circuit to be relaxed [ie., No initial inductor current].

At $t = 0^+$, the current must still be zero, since the current through an inductor cannot become zero instantaneously, even if it is not zero at $t = 0^-$.

$$\therefore i = 0 \quad At\ t = 0^+$$

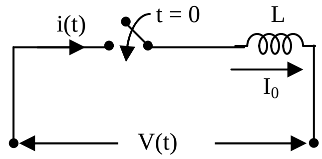
Hence, it is obvious that at $t(0^+)$, the inductor 'L' act as an open circuit. The equivalent circuit at $t(0^+)$ is as shown in figure…



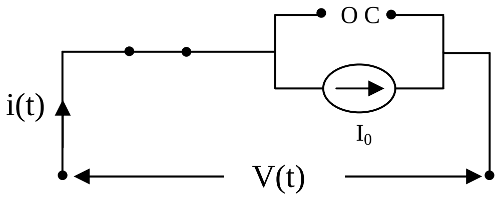
$\Rightarrow$ 'L' is replaced by an open circuit

However, if at $t(0^-)$ the inductor is already carrying a current due to a previously applied forcing function, it would continue to flow at $t(0^+)$, without change of magnitude.

Let 'I_0' be the initial inductor current as shown in figure…



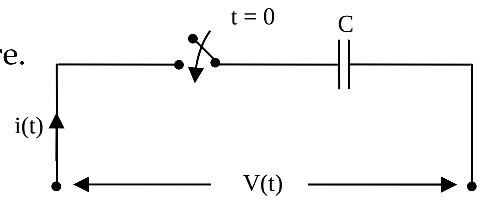
The equivalent circuit at $t(0^+)$ is shown in figure…



'L' is replaced by open and 'I_0' is replaced by equal current source.

CAPACITOR: -

If the circuit contains pure capacitor as shown in figure.
If the switch is closed at t = 0 by, applying
KVL to the circuit



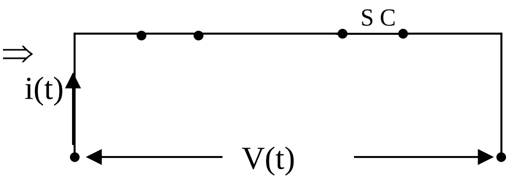
$$V = \frac{1}{C}\int i\,dt \quad for \quad t > 0^+$$

$$i = C\frac{dV}{dt}$$

At $t = 0^-$, the voltage across capacitor is zero, assuming the circuit to be relaxed [ie., No initial capacitor voltage]. Also current, $i = 0$.

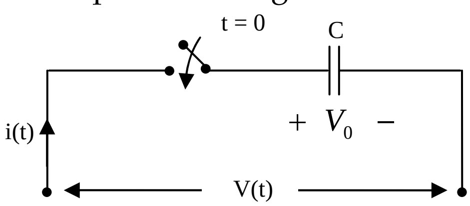
At $t = 0^+$, the voltage across capacitor must be zero, since the voltage across a capacitor cannot become zero instantaneously even if it is not zero at $t = 0^-$. It means that, at $t = 0^+$, the capacitor acts as Short Circuit. The equivalent circuit at $t(0^+)$ is as shown in figure…

Hence 'C' is replaced by short circuit $\Rightarrow$

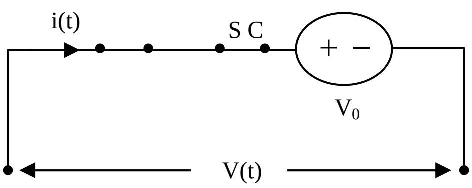


However, at $t = 0^-$, if there is a capacitor voltage due to previously applied forcing function, then at $t = 0^+$ also, it would remain without change in magnitude.

Let 'V_0' be initial capacitor voltage as shown in figure…



The equivalent circuit at $t(0^+)$ is shown in figure…



'C' is replaced by Short Circuit and 'V_0' is replaced by an equal voltage source.

FINAL VALUES OF NETWORK ELEMENTS

We shall next see how we can obtain equivalent circuits under Steady state conditions ie., at t = ∞ .

RESISTOR: -

A resistor obviously remains un effected. Hence a resistance R of a given network remains as 'R' only in the equivalent circuit at t = ∞. also.

INDUCTOR: -

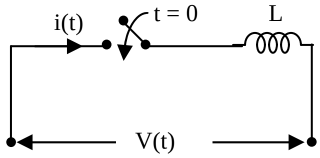
We have induced e.m.f in an inductor across 'L' then

$$V_L = L\frac{di}{dt}$$

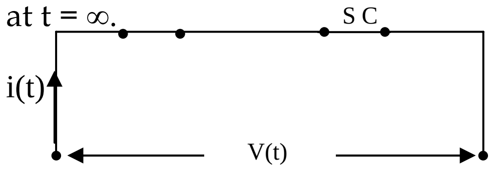
When Steady state has been reached ie., at t = ∞., there is no change of current

$$\text{i.e., } \frac{di}{dt} = 0 \qquad \therefore V_L = 0$$

Since, there is no inductor voltage, it implies that the inductor acts as short – circuit. Hence an inductor acts as open – circuit at t = 0^+, but it acts as short – circuit at t = ∞.



Equivalent circuit at t = ∞.



CAPACITOR: -

The current through a capacitor is given as

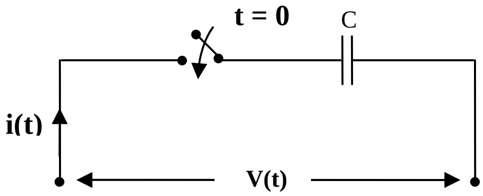
$$i = C\frac{dV_C}{dt} \text{ where } V_C \longrightarrow \text{Capacitor voltage}$$

At Steady state, there is no change of capacitor voltage

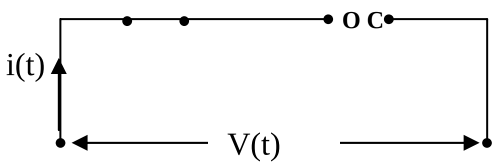
$$\frac{dV_C}{dt} = 0 \qquad \therefore i = 0$$

It implies that the capacitor acts as an open – circuit at t = ∞.

Hence a capacitor acts as short – circuit at t = 0^+, but acts as open – circuit at t = ∞.



Equivalent circuit of t = ∞.



SUMMARY:

- The current through an inductor cannot change instantaneously.
- Voltage across a capacitor cannot change instantaneously.
- At t (0^+), an inductor acts as an open – circuit.
- At t (0^+), a capacitor acts as short – circuit.
- At t (0^+), with initial inductor current 'I_0' is replaced by an equal current source with the same polarity.
- At t (0^+), with an initial capacitor voltage 'V_0' is replaced by an equal voltage source with the same polarity.
- At t (0^+) and at t = ∞, a resistor remains as it is, without any change.

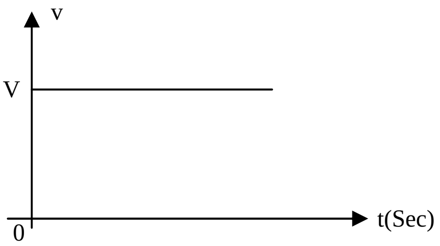
- ❖ An inductor acts as short – circuit, under Steady state conditions, for a forcing function of constant magnitude like step (or) DC voltage.
- ❖ A capacitor acts as open – circuit, under Steady state conditions, for a forcing function of constant magnitude like step (or) DC voltage.

TRANSIENT RESPONSE IN TIME DOMAIN WITH CONSTANT INPUT [DC EXCITATION]

RC CIRCUIT

The constant input as shown in figure….. is called step input (or) constant input. Since it steps from 0 to V volts at a time t=0

Let us assume that the voltage is suddenly applied at t=0 to the RC circuit shown in figure. Let us assume the initial charge on the capacitor is zero.



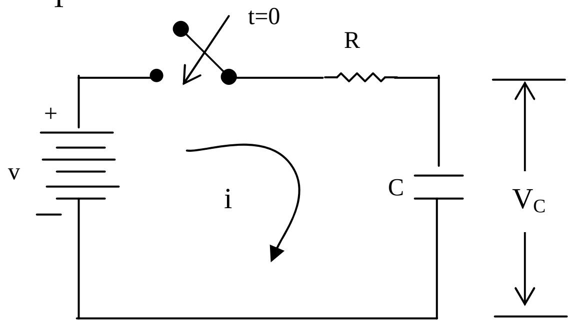


FIG : RC Circuit with Step Input

$\because$ At $t = 0^-$ ⟶ $V_C(O^-) = 0$

At $t = 0^+$ ⟶ $V_C(O^+) = 0$ [$\because$ Voltage across capacitor cannot change instantaneously]

Initially it act as short circuit

$$\therefore i(O^+) = \frac{V}{R}$$

Let 'i' be the current flowing in the circuit when the switch is closed at t = 0. Using KVL, the equilibrium equation is

$$Ri + \frac{1}{C}\int i\,dt = v \qquad ....(1)$$

Differentiating the equation (1) with respect to time 't'.

$$R\frac{di}{dt} + \frac{1}{C}i = 0 \Rightarrow \frac{di}{dt} + \frac{1}{RC}i = 0 \;\Rightarrow \left[D + \frac{1}{RC}\right]i = 0 \qquad ....(2)$$

The equation (2) is a first order linear homogeneous equation. Hence the total solution will have only complementary function, and particular integral is zero.

$$i(t) = Ae^{-t/RC} \qquad .... (3)$$

To evaluate the constant 'A' we will use the initial condition i.e.,

$$i(0^+) = \frac{V}{R}$$

$$i(0)\,|_{t=0} = A = \frac{V}{R}$$

$$\therefore i(t) = \frac{V}{R}e^{-t/RC} \; (or)\, i(t) = \frac{V}{R}e^{-t/\tau} \qquad ....(4)$$

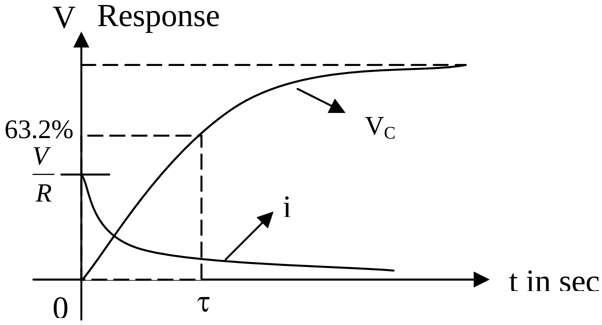
This solution is called Natural response of the circuit and also called as the complementary function. Where τ=RC is called the time constant of an RC circuit.

VOLTAGE ACROSS CAPACITOR:

$$V_C = \frac{1}{C}\int i\,dt = \frac{1}{C}\int_0^t \frac{V}{R}e^{-t/\tau}dt = \frac{V}{RC}\frac{1}{\left[\frac{-1}{\tau}\right]}\left[e^{-t/\tau}\right]_0^t$$

$$\frac{V}{RC}(-RC)\left[e^{-t/\tau}\right]_0^t = -V\left[e^{-t/\tau}-1\right] = V\left[1-e^{-t/\tau}\right] \quad .....(5)$$

As 't' is varying from 0 to ∞ the time response characteristics of current and voltage across the capacitor from equations (4) and (5) are shown in figure.



The transient solution is a total solution of the circuits i.e., $i(t)=i_{ss}+i_t = \frac{V}{R}e^{-t/\tau}$. Where '$i_{ss}$' is steady state value and 'i_t' is the transient value. The response of the circuit will depend upon 'τ'. If 'R' and 'C' are larger then the circuit takes longer time to settle down to the new steady state value.

TIME CONSTANT

The interpretation of the time constant as 'the time interval during which the response of the circuit starting from any point of time during the transient interval, would have reached its final value if it had maintained its rate of change constant at the value it had at that point of time". If the time is equal to one time constant then $V_C = V(1-e^{-1}) = 0.632V$. ***The time constant can be regarded as the time required for the transient response to attain 63.2% of the steady state value starting from zero***. In two, three, four (or) five time constants the time response values would be 0.864, 0.95, 0.982 and 0.993 of its steady state value. For all practical purposes most of the electrical instruments used for measurement of electrical quantities will have a least count of 1% after approximately five time constants have elapsed.

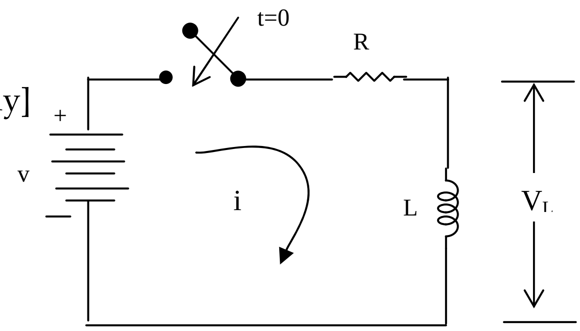
RL CIRCUIT

The RL network is excited by a step input is shown in figure. Let us assume that at the time t=0 the switch is closed and initially the current through the inductor is zero.

At $t = 0^-$ ⟶ $i_L(0^-) = 0$

At $t = 0^+$ ⟶ $i_L(0^+) = 0$

[∵ Current in the inductor cannot change instantaneously]



Using KVL, the equilibrium equation is

$$Ri + L\frac{di}{dt} = v \quad ....(1)$$

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L} \Rightarrow \left[D + \frac{R}{L}\right]i = \frac{V}{L} \quad ....(2)$$

The equation (2) is a first order differential equation and the solution gives the response of the circuit. To get the solution we will obtain the transient part [complementary function] and steady state part [particular integral] separately. The transient part of the solution is obtained by solving the homogenous part of the differential equation by making forcing function to zero.

i.e., $\left[D + \frac{R}{L}\right]i = 0$

The general solution is of the form

$$i_t = Ae^{-\frac{R}{L}t} \quad .....(3)$$

Steady state part of the solution (or) particular integral is obtained from

$$\left[D+\frac{R}{L}\right]i=\frac{V}{L}\Rightarrow i=\frac{V}{L\left[D+\frac{R}{L}\right]}$$

To get the steady state part of the solution, substitute D=0 [for DC excitation].

$$i_{ss}=\frac{V}{L*\frac{R}{L}}=\frac{V}{R} \qquad \ldots.(4)$$

The complete solution is

$$i=i_{ss}+i_t=\frac{V}{R}+Ae^{-\frac{R}{L}t}$$

To evaluate the constant A, we use initial condition $i(0^+)=0$ At $t = 0^+$

$$i(0^+)=\frac{V}{R}+A\Rightarrow 0=A+\frac{V}{R}\Rightarrow A=-\frac{V}{R}$$

$\therefore$ The complete solution is

$$i(t)=\frac{V}{R}-\frac{V}{R}e^{-\frac{R}{L}t}=\frac{V}{R}\left[1-e^{-t/\tau}\right] \qquad \ldots.(5)$$

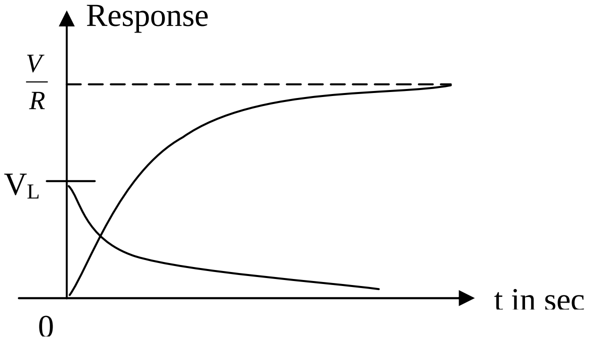
Where 'τ' is the time constant and is equal to $\frac{L}{R}$

VOLTAGE ACROSS THE INDUCTOR:

The voltage across the inductor $V_L=L\frac{di}{dt}$

$$\Rightarrow V_L=L\frac{d\left[\frac{V}{R}\left(1-e^{-t/\tau}\right)\right]}{dt}=\frac{VL}{R}\left(\frac{1}{\tau}\right)e^{-t/\tau}=\frac{VL}{R}*\frac{R}{L}e^{-t/\tau}=Ve^{-t/\tau} \qquad \ldots\ldots.(6)$$

As 't' is varying from 0 to ∞ the time response characteristics of current and voltage across the inductor from equations (5) & (6) are shown in figure……

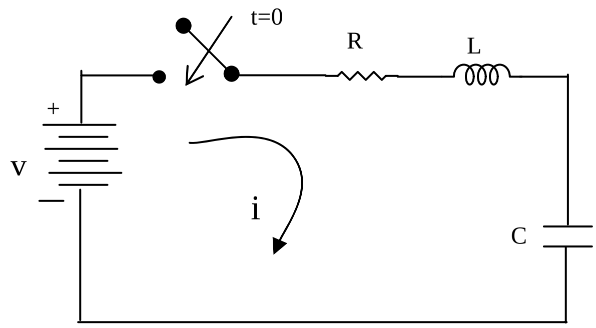


TIME CONSTANT

If the time is equal to one time constant then $V_L = V(e^{-1}) = 0.3678$ Volts. The time constant can be regarded as the time required for the transient response to attain 36.78% of initial value.

RLC CIRCUIT

The behaviour of an RLC series circuit with constant excitation is presented here. Such RLC circuits are of great importance, since they occur, in many practical situations. In the figure shown above. A battery of voltage 'V' is suddenly applied to the series RLC circuit with no-initial current in the inductor and initial charge on the capacitor



At $t = 0^-$ ⟶ $i_L(0^-) = 0$; $V_C(0^-) = 0$

At $t = 0^+$ ⟶ $i_L(0^+) = 0$; $V_C(0^+) = 0$

Applying KVL, the equilibrium equation is

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = V \qquad \text{....(1)}$$

Differentiating with respect to time, 't'

$$R\frac{di}{dt} + L\frac{di^2}{dt^2} + \frac{1}{C}i = 0 \Rightarrow \frac{di^2}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0 \Rightarrow \left[D^2 + \frac{R}{L}D + \frac{1}{LC}\right]i = 0 \qquad \text{....(2)}$$

This is a second order differential equation and it is a homogeneous equation.
The solution of this equation is of the form

$$i = Ae^{m_1 t} + Be^{m_2 t} \qquad \text{.....(3)}$$

Where 'A' and 'B' are constants to be determined from the initial conditions of the network and m_1 and m_2 are the roots of characteristic equations.

$$D^2 + \frac{R}{L}D + \frac{1}{LC} = 0$$

The roots of the characteristic equation are

$$m_1, m_2 = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - 4\left(\frac{1}{LC}\right)}}{2} = \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

The response of the network depends on the nature of the roots m_1 and m_2. Also depend up on the value under radical. Three cases of these roots are explained below.

CASE 1 : When $\sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$ is positive. In this case $\left(\frac{R}{2L}\right)^2 > \frac{1}{LC}$. Hence the roots are negative real. The response of the circuit is with out oscillations as shown in figure, curve 1. In this case the final value is reached more slowly and is said to be over damped.

CASE 2 : When $\sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$ is equal to zero. In this case $\left(\frac{R}{2L}\right)^2 = \frac{1}{LC}$. Hence the roots are equal to $-\frac{R}{2L}$. In this case the response rises faster than curve 1 without any oscillations and no over-shoot on the final value. This response is called critically damped and is shown by curve 2 in figure. The time of response is shortest.

CASE 3 : When $\sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$ is negative i.e., $\left(\frac{R}{2L}\right)^2 < \frac{1}{LC}$ then the roots m_1 and m_2 are complex conjugates with negative real parts. The response of the system is oscillatory with over shoots on the final value. This response is termed as under damped. Such a response is often said to be ringing. The under damped behaviour is shown by curve 3 in figure.

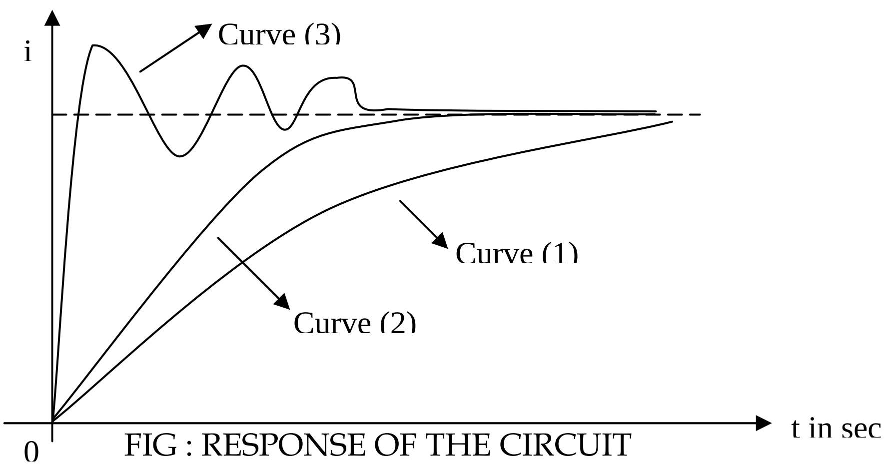


FIG : RESPONSE OF THE CIRCUIT

THE EXPRESSION FOR CIRCUIT CURRENT

The expression for current of an RLC series circuit may be written as

$$\frac{d^2i}{dt^2} + 2\zeta\omega_n \frac{di}{dt} + \omega_n^2 i = 0$$

Whose roots are $m_1, m_2 = -\zeta\,\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$ and the solution is

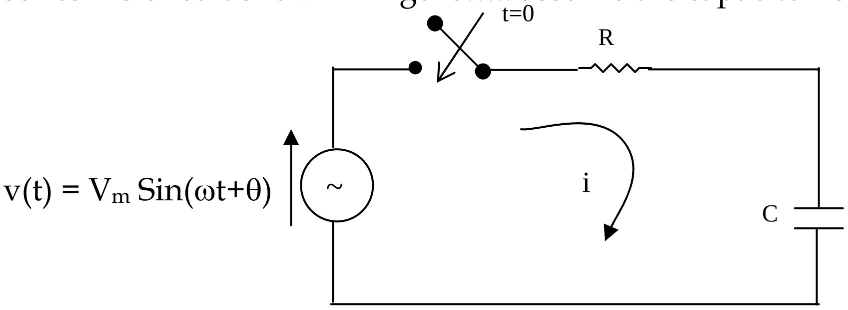
$$i(t) = C_1 e^{\left[-\zeta\,\omega_n + \omega_n\sqrt{\zeta^2-1}\right]t} + C_2 e^{\left[-\zeta\,\omega_n - \omega_n\sqrt{\zeta^2-1}\right]t}$$

The solution will have different farms depending up on the value of 'ζ'. If 'ζ' is less than one then $R<R_C$ and the response is under damped. If 'ζ' is equal to one then $R=R_C$ and the response is critically damped. If 'ζ' is more than one then $R>R_C$ and the response is over damped.

TRANSIENT RESPONSE WITH SINUSOIDAL INPUT [AC EXCITATION]

RC CIRCUIT

Let a sinusoidal voltage i.e., $v(t) = V_m \sin(\omega t+\theta)$ be suddenly applied at time t=0 to the series RC circuit shown in figure….. assume the capacitor is initially unchanged.



At $t = 0^-$ ⟶ $V(0^-) = 0$

At $t = 0^+$ ⟶ $V(0^+) = 0$

And V_m and θ are constants. Applying KVL, the equilibrium equation is

$$Ri + \frac{1}{C}\int i\,dt = V_m \, Sin(\omega t + \theta) \qquad ….(1)$$

Differentiating with respect to time, 't'

$$R\frac{di}{dt} + \frac{1}{C}i = V_m\omega Cos(\omega t + \theta) \qquad …..(2)$$

$$\Rightarrow \frac{di}{dt} + \frac{1}{RC}i = \frac{V_m\omega}{R}Cos(\omega t + \theta)$$

$$\Rightarrow \left[D + \frac{1}{RC}\right]i = \frac{V_m\omega}{R}Cos(\omega t + \theta) \qquad ….(3)$$

The complementary function (or) transient part of the above differential equation is

$$i_t = K_1 e^{-t/RC} = K_1 e^{-t/\tau} \qquad ….(4)$$

Where τ=RC ⟶ Time constant

The steady state part of the solution of equation (1) is as follows. Let us assume that the particular integral is of the form

$$i_p (or) i_{ss} = ACos(\omega t + \theta) + B\,Sin(\omega t + \theta)$$

Where $\frac{di_p}{dt} = -ASin(\omega t + \theta)\omega + BCos(\omega t + \theta)\omega$

$= -A\omega\,Sin(\omega t + \theta) + B\omega\,Cos(\omega t + \theta)$

Substituting in equation (2) we have

$$R[-A\omega \, Sin(\omega t+\theta) + B\omega Cos(\omega t+\theta)] + \frac{1}{C}\left[ACos(\omega t+\theta) + BSin(\omega t+\theta)\right]$$

$$= V_m \, \omega Cos(\omega t+\theta)$$

$$\left[\frac{B}{C} - AR\omega\right] Sin(\omega t+\theta) + \left[BR\omega + \frac{A}{C}\right] Cos(\omega t+\theta) = V_m \omega \, Cos(\omega t+\theta)$$

Equating L.H.S. and R.H.S. we have

$$\frac{B}{C} - AR\omega = 0 \Rightarrow AR\omega = \frac{B}{C} \Rightarrow A = \frac{B}{RC\omega} \qquad \ldots.(5)$$

$$BR\omega + \frac{A}{C} = V_m \omega \Rightarrow BR\omega + \frac{B}{RC^2\omega} = V_m \omega \;[\because \; from\, equation\,(5)]$$

$$\Rightarrow B\left[\frac{R^2C^2\omega^2 + 1}{RC^2\omega}\right] = V_m \omega \Rightarrow B = \frac{V_m RC^2\omega^2}{(R^2C^2\omega^2 + 1)} \qquad \text{(or)} \;\; B = \frac{V_m R}{\left[R^2 + \frac{1}{C^2\omega^2}\right]}$$

Now, $A = \dfrac{V_m R}{\left[R^2 + \frac{1}{C^2\omega^2}\right]} * \dfrac{1}{RC\omega} = \dfrac{V_m}{C\omega\left[R^2 + \frac{1}{C^2\omega^2}\right]}$

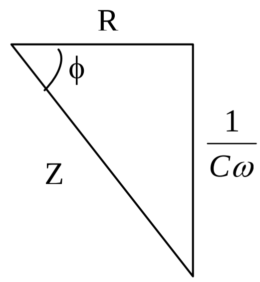
Then steady state solution is given by

$$i_{ss} = \frac{V_m}{C\omega\left[R^2 + \frac{1}{C^2\omega^2}\right]} Cos(\omega t+\theta) + \frac{V_m R}{\left[R^2 + \frac{1}{C^2\omega^2}\right]} Sin(\omega t+\theta)$$

$$= \frac{V_m}{\left[R^2 + \frac{1}{C^2\omega^2}\right]}\left[\frac{Cos(\omega t+\theta}{C\omega} + R\,Sin(\omega t+\theta)\right] \qquad \ldots\ldots(6)$$

The above expression can be written as single sinusoidal function with phase angle ‘ϕ’. The equation (6) can be written as

$$i_{ss} = \frac{V_m}{\sqrt{\left(R^2 + \frac{1}{C^2\omega^2}\right)}}\left[\frac{Cos(\omega t+\theta)}{C\omega\sqrt{\left(R^2 + \frac{1}{C^2\omega^2}\right)}} + \frac{R\,Sin(\omega t+\theta)}{\sqrt{\left(R^2 + \frac{1}{C^2\omega^2}\right)}}\right] \qquad \ldots\ldots(7)$$



From the impedance triangle of the circuit shown in figure, we get

$$Cos\phi = \frac{R}{\sqrt{\left(R^2 + \frac{1}{C^2\omega^2}\right)}}; \; Sin\phi = \frac{1/C\omega}{\sqrt{\left(R^2 + \frac{1}{C^2\omega^2}\right)}}$$

Hence equation (7) reduced to

$$i_{ss} = \frac{V_m}{\sqrt{\left(R^2 + \frac{1}{C^2\omega^2}\right)}}\left[Sin\phi\, Cos(\omega t+\theta) + Cos\phi\, Sin(\omega t+\theta)\right]$$

$$i_{ss} = \frac{V_m}{\sqrt{\left(R^2 + \frac{1}{C^2\omega^2}\right)}} Sin(\omega t+\theta+\phi) \qquad \ldots\ldots..(8)$$

Where $\phi = Tan^{-1}\dfrac{1}{\omega CR}$ ⟶ Impedance angle of the circuit.

The equation (8) is similar to $\frac{V_m}{Z} Sin(\omega t+\theta+\phi)$ which is same as steady state value of current in a RC circuit.

i.e., $i_{ss} = \frac{V_m}{Z} Sin(\omega t+\theta+\phi)$ Leads the voltage by angle 'ϕ'.

The complete solution is

$$i(t) = Ke^{-t/\tau} + \frac{V_m}{Z} Sin(\omega t+\theta+\phi) \qquad ....(9)$$

The constant 'K' is evaluated using initial condition i.e., from equation (1) and substituting $V_C(O^+) = 0$

At $t = 0^+$; $Ri = V_m \, Sin\theta \qquad \Rightarrow i(0^+) = \frac{V_m}{R} Sin\theta$

Substituting initial conditions in equation (9) we get

$$\frac{V_m}{R} Sin\theta = K_1 + \frac{V_m}{Z} Sin(0+\theta+\phi)$$

Where $Z = \sqrt{\left(R^2 + \frac{1}{C^s\omega^2}\right)}$ & $\phi = Tan^{-1}\frac{1}{\omega CR}$

$$\therefore K = \frac{V_m}{R} Sin\theta - \frac{V_m}{Z} Sin(\theta+\phi)$$

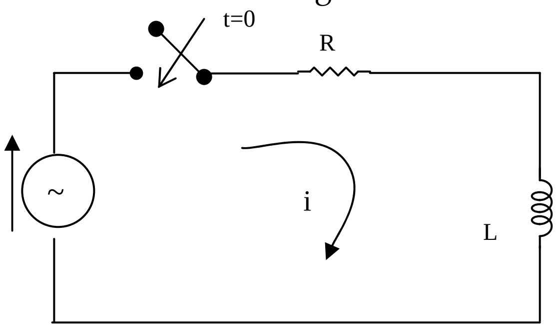
$\therefore$ The complete solution is given by

$$i(t) = \left[\frac{V_m}{R} Sin\theta - \frac{V_m}{Z} Sin(\theta+\phi)\right] e^{-t/\tau} + \frac{V_m}{Z} Sin(\omega t+\theta+\phi) \qquad ....(10)$$

RL CIRCUIT

Consider a RL Circuit is shown in figure. Let us assume there was no initial current in the inductor.

v(t) = V_m Sin(ωt+θ)



At $t = 0^-$ ⟶ $i_L(0^-) = 0$
At $t = 0^+$ ⟶ $i_L(0^+) = 0$

Applying KVL, the equilibrium equation is

$$Ri + L\frac{di}{dt} = V_m \, Sin(\omega t+\theta) \qquad ....(1)$$

$$\Rightarrow \frac{di}{dt} + \frac{R}{L}i = \frac{V_m}{L} Sin(\omega t+\theta)$$

$$\Rightarrow \left[D + \frac{R}{L}\right] i = \frac{V_m}{L} Sin(\omega t+\theta) \qquad ....(2)$$

The complementary function (or) transient part of the above differential equation is

$$i_t = K_2 e^{-\frac{R}{L}t} = K_2 e^{-t/\tau} \qquad ....(3)$$

Where $\tau = \frac{L}{R}$ ⟶ Time constant

The steady state part of the solution of equation (1) as follows. Let us assume that the particular integral is of the form

$$i_p = A Cos(\omega t+\theta) + B\, Sin(\omega t+\theta)$$

Then $\frac{di_p}{dt} = -A\omega\, Sin(\omega t+\theta) + B\omega\, Cos(\omega t+\theta)$

Substituting in equation (1) we have

R [A Cos $(\omega t+\theta)$ + B Sin$(\omega t+\theta)$] + L [-Aω Sin$(\omega t+\theta)$ +Bω Cos$(\omega t+\theta)$]

= V_m Sin$(\omega t+\theta)$

(BR-ALω) Sin $(\omega t+\theta)$ + (AR+BLω) Cos $(\omega t+\theta)$ = V_m Sin $(\omega t+\theta)$

Equating L.H.S. and R.H.S. we have

AR + BLω = 0 ⇒ A = $-\frac{BL\omega}{R}$

$$BR - AL\omega = V_m \Rightarrow BR + \frac{BL^2\omega^2}{R} = V_m \Rightarrow B\left[\frac{R^2+L^2\omega^2}{R}\right] = V_m \text{ (or) } B = \frac{V_m R}{(R^2+L^2\omega^2)}$$

$$\therefore A = -\frac{V_m RL\omega}{R(R^2+L^2\omega^w)} = \frac{V_m L\omega}{(R^2+L^2\omega^2)}$$

∴ The steady state solution is given by

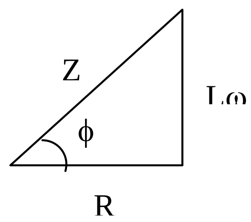
$$i_{ss} = \frac{V_m}{(R^2+L^2\omega^2)}\left[R\, Sin(\omega t+\theta) - \omega L\, Cos(\omega t+\theta)\right] \qquad ....(4)$$

The above expression can be written as single sinusoidal function with phase angle 'ϕ'.
The equation (4) can be written as

$$i_{ss} = \frac{V_m}{\sqrt{(R^2+L^2\omega^2)}}\left[\frac{R}{\sqrt{(R^2+L^2\omega^2)}} Sin(\omega t+\theta) - \frac{L\omega}{\sqrt{(R^2+L^2\omega^2)}} Cos(\omega t+\theta)\right] \qquad ...(5)$$

From the impedance triangle of the circuit shown in figure we get

$$Cos\phi = \frac{R}{\sqrt{(R^2+L^2\omega^2)}};\ Sin\phi = \frac{L\omega}{\sqrt{(R^2+L^2\omega^2)}}$$



Hence the equation (5) is reduced to

$$i_{ss} = \frac{V_m}{\sqrt{(R^2+L^2\omega^2)}}\left[Cos\phi\ Sin(\omega t+\theta) - Sin\phi\ Cos(\omega t+\theta)\right]$$

$$i_{ss} = \frac{V_m}{\sqrt{(R^2+L^2\omega^2)}} Sin(\omega t+\theta-\phi) \qquad .....(6)$$

Where $\phi = Tan^{-1}\frac{\omega L}{R}$ ⟶ Impedance angle of the circuit.

The equation (6) is similar to $\frac{V_m}{Z}$Sin(ωt+θ-ϕ) which is same as steady state value of current in a RL circuit.

$i_{ss} = \frac{V_m}{Z} Sin(\omega t+\theta-\phi)$ Lags the voltage by an angle ϕ.

The complete solution is

$$i = K_2 e^{\frac{-Rt}{L}} + \frac{V_m}{Z} Sin(\omega t+\theta-\phi) \qquad ....(7)$$

The constant K_2 is to be evaluated using initial condition i.e., $i(0^+) = 0$ in equation (7)

$$0 = K_2 + \frac{V_m}{Z} Sin(0+\theta-\phi) \Rightarrow K_2 = -\frac{V_m}{Z} Sin(\theta-\phi) \qquad .....(8)$$

$\therefore$ The complete solution is

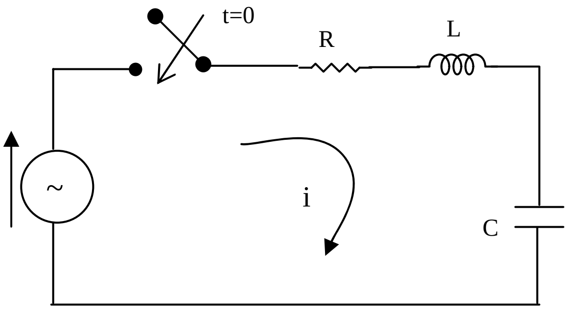
$$i(t) = \frac{V_m}{Z}\left[Sin(\omega t+\theta-\phi) - Sin(\theta-\phi)e^{\frac{-Rt}{L}} \right] \text{(or)} = \frac{V_m}{Z}\left[Sin(\omega t+\theta-\phi) - Sin(\theta-\phi)e^{-t/\tau} \right] .....(9)$$

In equation (9) the first term gives the steady state response of the current which is sinusoidal and Lagging the applied voltage by an angle $\phi = Tan^{-1}\frac{\omega L}{R}$. The second part gives the transient response which is reduced to zero in short time.

RLC CIRCUIT

Consider a series RLC circuit as shown in figure. Let us assume that no-initial current in the inductor and no-initial charge in the capacitor.

v(t) = V_m Sin(ωt+θ)



At t = 0^- ⟶ $i_L(0^-) = 0; V_C(0^-) = 0$

At t = 0^+ ⟶ $i_L(0^+) = 0; V_C(0^+) = 0$

Applying KVL, the equilibrium equation is

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = V_m\, Sin(\omega t+\theta) \qquad ....(1)$$

Differentiating with respect to time, 't'

$$R\frac{di}{dt} + L\frac{di^2}{dt^2} + \frac{1}{C}i = V_m\omega\; Cos(\omega t+\theta) \qquad .....(2)$$

$$\Rightarrow \frac{di^2}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = \frac{V_m\omega}{L}Cos(\omega t+\theta)$$

$$\Rightarrow \left[D^2 + \frac{R}{L}D + \frac{1}{LC}\right]i = \frac{V_m\omega}{L}Cos(\omega t+\theta) \qquad ....(3)$$

The steady state (or) particular solution can be obtained as follows :

Let $i_p = ACos(\omega t+\theta) + B\,Sin(\omega t+\theta)$

Then $\frac{di_p}{dt} = -A\omega\, Sin(\omega t+\theta) + B\omega\, Cos(\omega t+\theta)$

$$\frac{d{i_p}^2}{dt^2} = -A\omega^2\, Cos(\omega t+\theta) - B\omega^2 Sin(\omega t+\theta)$$

Substituting in equation (2) we have

$$\frac{1}{C}[A\ Cos\ (\omega t+\theta) + B\ Sin(\omega t+\theta)] + R\ [-A\omega\ Sin(\omega t+\theta) + B\omega\ Cos(\omega t+\theta)]$$

$$+ L\ [-A\omega^2\ Cos(\omega t+\theta) - B\omega^2\ Sin(\omega t+\theta)] = V_m\omega\ Cos\ (\omega t+\theta)$$

$$\Rightarrow \left[\frac{B}{C} - AR\omega - BL\omega^2\right]Sin(\omega t+\theta) + \left[\frac{A}{C} + BR\omega - AL\omega^2\right]Cos(\omega t+\theta) = V_m\omega Cos(\omega t+\theta)$$

Equating L.H.S. and R.H.S. we have

$$\frac{B}{C} - AR\omega - BL\omega^2 = 0 \Rightarrow -B\omega^2 - \frac{AR\omega}{L} + \frac{1}{LC} = 0$$

$$\Rightarrow -A\left(\frac{\omega R}{L}\right) - B\left[\omega^2 - \frac{1}{LC}\right] = 0 \Rightarrow A\left(\frac{\omega R}{L}\right) + B\left(\omega^2 - \frac{1}{LC}\right) = 0$$

$$\frac{A}{C} + BR\omega - AL\omega^2 = V_m\omega \Rightarrow -A\omega^2 + \frac{BR\omega}{L} + \frac{A}{LC} = \frac{V_m\omega}{L} \Rightarrow -A\left[\omega^2 - \frac{1}{LC}\right] + B\left(\frac{R\omega}{L}\right) = \frac{V_m\omega}{L}$$

$$\because A = -B\frac{\left[\omega^2 - \frac{1}{LC}\right]}{\left(\frac{\omega R}{L}\right)} = B\frac{\left[-\omega^2 + \frac{1}{LC}\right]}{\left(\frac{\omega R}{L}\right)} \Rightarrow B\left\{\left[-\omega^2 + \frac{1}{LC}\right]^2 + \frac{\omega^2 R^2}{L^2}\right\} = \frac{\omega^2 R V_m}{L^2}$$

$$\therefore B = \frac{V_m\left(\frac{R\omega^2}{L^2}\right)}{\left\{\left[\frac{1}{LC} - \omega^2\right]^2 + \frac{\omega^2R^2}{L^2}\right\}} \quad \& \quad \therefore A = B\frac{\left[-\omega^2 + \frac{1}{LC}\right]}{\left(\frac{\omega R}{L}\right)} = \frac{V_m\omega\left[\frac{1}{LC} - \omega^2\right]}{L\left\{\left(\frac{1}{LC} - \omega^2\right)^2 + \frac{\omega^2R^2}{L^2}\right\}}$$

$$i_p = \frac{V_m\omega\left[\frac{1}{LC} - \omega^2\right]}{L\left\{\left(\frac{1}{LC} - \omega^2\right)^2 + \frac{\omega^2R^2}{L^2}\right\}}\mathrm{Cos}(\omega t + \theta) + \frac{V_m\left(\frac{R\omega^2}{L^2}\right)}{\left\{\left(\frac{1}{LC} - \omega^2\right)^2 + \frac{\omega^2R^2}{L^2}\right\}}\mathrm{Sin}(\omega t + \theta)$$

$$\text{Let } M\,\mathrm{Sin}\phi = \frac{V_m\omega\left[\frac{1}{LC} - \omega^2\right]}{L\left\{\left(\frac{1}{LC} - \omega^2\right)^2 + \frac{\omega^2R^2}{L^2}\right\}}; \quad M\,\mathrm{Cos}\phi = \frac{V_m\left(\frac{R\omega^2}{L^2}\right)}{\left\{\left(\frac{1}{LC} - \omega^2\right)^2 + \frac{\omega^2R^2}{L^2}\right\}}$$

$$\text{Then } \frac{M\,\mathrm{Sin}\phi}{M\,\mathrm{Cos}\phi} = \mathrm{Tan}\phi = \frac{V_m\omega\left[\frac{1}{LC} - \omega^2\right]}{L\left\{\left(\frac{1}{LC} - \omega^2\right)^2 + \frac{\omega^2R^2}{L^2}\right\}} * \frac{\left\{\left(\frac{1}{LC} - \omega^2\right)^2 + \frac{\omega^2R^2}{L^2}\right\}}{V_m\left(\frac{R\omega^2}{L^2}\right)}$$

$$= \frac{\omega}{L} * \left(\frac{1}{LC} - \omega^2\right) * \frac{L^2}{R\omega^2} = \left(\frac{1}{LC} - \omega^2\right)\frac{L}{\omega R} = \frac{1}{R}\left(\frac{L}{LC\omega} - \frac{\omega^2 L}{\omega}\right) = \frac{1}{R}\left(\frac{1}{C\omega} - L\omega\right)$$

$$\therefore \phi = \mathrm{Tan}^{-1}\left\{\frac{1}{R}\left[\frac{1}{C\omega} - L\omega\right]\right\}$$

$$\therefore i_p = M\,\mathrm{Sin}\phi\,\mathrm{Cos}(\omega t + \theta) + M\,\mathrm{Cos}\phi\,\mathrm{Sin}(\omega t + \theta) = M\,\mathrm{Sin}\,(\omega t + \theta + \phi)$$

Where $\phi = \mathrm{Tan}^{-1}\left\{\frac{1}{R}\left[\frac{1}{C\omega} - L\omega\right]\right\} \longrightarrow$ Impedance angle of the circuit.

However, $M = \sqrt{M^2 Cos^2\phi + M^2 Sin^2\phi}$

$$= \sqrt{\frac{V_m^{\ 2}\omega^2\left(\frac{1}{LC} - \omega^2\right)^2}{L^2\left\{\left(\frac{1}{LC} - \omega^2\right)^2 + \frac{\omega^2R^2}{L^2}\right\}^2} + \frac{V_m^2\left(\frac{R\omega^2}{L^2}\right)^2}{\left\{\left(\frac{1}{LC} - \omega^2\right)^2 + \frac{\omega^2R^2}{L^2}\right\}^2}} = \sqrt{\frac{V_m^{\ 2}\omega^2\left\{\left(\frac{1}{LC} - \omega^2\right)^2 + R^2\right\}}{L^2\left\{\left(\frac{1}{LC} - \omega^2\right)^2 + \frac{\omega^2R^2}{L^2}\right\}^2}}$$

$$= \sqrt{\frac{V_m^{\ 2}\omega^2}{L^2\left\{\left(\frac{1}{LC} - \omega^2\right)^2 + \frac{\omega^2R^2}{L^2}\right\}}} = \sqrt{\frac{V_m^{\ 2}}{\frac{L^2}{\omega^2}\left\{\left(\frac{1}{LC} - \omega^2\right)^2 + \frac{\omega^2R^2}{L^2}\right\}}}$$

If $\left(\frac{1}{C\omega} - L\omega\right)^2 = \frac{1}{C^2\omega^2} + L^2\omega^2 - 2\frac{L\omega}{C\omega} = \frac{1}{C^2\omega^2} + L^2\omega^2 - 2\frac{L}{C}$

If $\frac{L^2}{\omega^2}\left[\left(\frac{1}{LC} - \omega^2\right)^2\right] = \frac{L^2}{\omega^2}\left[\frac{1}{L^2C^2} + \omega^4 - \frac{2\omega^2}{LC}\right] = \frac{L^2}{L^2C^2\omega^2} + \frac{\omega^4 L^2}{\omega^2} - \frac{2\omega^2 L^2}{LC\omega^2}$

$= \frac{1}{C^2\omega^2} + L^2\omega^2 - \frac{2L}{C} = \left(\frac{1}{C\omega} - L\omega\right)^2$

$$\therefore M = \sqrt{\frac{V_m^2}{R^2 + \left(\frac{1}{C\omega} - L\omega\right)^2}} = \frac{V_m}{\sqrt{R^2 + \left(\frac{1}{C\omega} - L\omega\right)^2}}$$

Thus $i_p = \frac{V_m}{\sqrt{R^2 + \left(\frac{1}{C\omega} - L\omega\right)^2}} \text{Sin}(\omega t + \theta + \phi)$

The complementary function being equal to the DC response of RLC circuit.

CASE 1 : Over damped, when $\left(\frac{R}{2L}\right)^2 > \frac{1}{LC}$

$$i = e^{\alpha t}(C_1 e^{\beta t} + C_2 e^{-\beta t}) + \frac{V_m}{\sqrt{R^2 + \left(\frac{1}{C\omega} - L\omega\right)^2}} \text{Sin}(\omega t + \theta + \phi)$$

CASE 2 : Critically damped, when $\left(\frac{R}{2L}\right)^2 = \frac{1}{LC}$

$$i = e^{\alpha t}(C_1 + C_2 t) + \frac{V_m}{\sqrt{R^2 + \left(\frac{1}{C\omega} - L\omega\right)^2}} \text{Sin}(\omega t + \theta + \phi)$$

CASE 3 : Under damped, when $\left(\frac{R}{2L}\right)^2 < \frac{1}{LC}$

$$i = e^{\alpha t}(C_1 \text{Cos}\beta t + C_2 \text{Sin}\beta t) + \frac{V_m}{\sqrt{R^2 + \left(\frac{1}{C\omega} - L\omega\right)^2}} \text{Sin}(\omega t + \theta + \phi)$$

THEOREMS FOR DETERMINATION OF INITIAL AND FINAL VALUES

Frequently it is desirable to determine the initial or final values of the response function before completing the solution of a problem. Even these values can be determined by inspection, the possibility of checking from the response transform is great value because the inverse transform of the response is tedious. Before finding the inverse transform it is good to find the initial and final values. These values can be found out by initial and final value theorems.

INITIAL VALUE THEOREM: -

If f (t) and its first derivative are laplace transformable, then the initial value of f (t) is

$$f(0^+) = \lim_{t\to 0} f(t) = \lim_{s\to\infty} s\,F(s)$$

PROOF:- Taking The First derivative of f(t)

$$\int_0^\infty \left[\frac{df(t)}{dt}\right] e^{-st}\,dt = s\,F(s) - f(0^+)$$

Now let S approaches ∞, we have

$$\lim_{S\to\infty} \left\{\left[\frac{df(t)}{dt}\right] e^{-st}\,dt\right\} = \lim_{S\to\infty} [s\,F(s) - f(0^+)]$$

Therefore, $0 = \lim_{S\to\infty} [s\,F(s) - f(0^+)]$

Therefore, $\lim_{S\to\infty} [s\,F(s)] = f(0^+)$

FINAL VALUE THEOREM:-

If f(t) and its first derivative are laplace transformable, then the final value of f(t) is

$$\lim_{t\to\infty} [f(t)] = \lim_{S\to 0} s\,F(s)$$

PROOF:- Taking the laplace transform on $\frac{df(t)}{dt}$. We have,

$$\int_0^\infty \left[\frac{df(t)}{dt}\right] e^{-st}\,dt = s\,F(s) - f(0^+)$$

Now let S approaches zero

$$\lim_{S\to 0} \int_0^\infty \left[\frac{d\,f(t)}{dt}\right] e^{-st}\,dt = \lim_{S\to 0} [s\,F(s) - f(0^+)] \qquad \text{................(1)}$$

$$\int_0^\infty \left[\frac{d\,f(t)}{dt}\right] dt = \lim_{t\to\infty} \left[\frac{d\,f(t)}{dt}\right] dt = \lim_{t\to\infty} [\,f(t) - f(0^+)\,] \qquad \text{................(2)}$$

From equations 1 & 2 we have

$$\lim_{t\to\infty} [\,f(t)\,] = \lim_{S\to 0} [\,sF(s)\,]$$